Indian J. Phys. **77B** (5), 529-532 (2003)

AN INTERNATIONAL JOURNAL
IJP B

Stability of superposed viscous-viscoelastic (Walters' B′) fluids in porous medium in hydrodynamics and hydromagnetics

R C Sharma* and Sanjeev Gangta

Department of Mathematics, Himachal Pradesh University, Shimla-171 005, India

E-mail : rcsharmashimla@hotmail.com

Received 26 December 2002, accepted 24 March 2003

Abstract The instability of the plane interface between two superposed viscous and viscoelastic (Walters' B') fluids in porous medium, in hydrodynamics and hydromagnetics, has been studied. The effect of a variable horizontal magnetic field is also considered separately. For the potentially stable arrangement, the system is stable or unstable according as $\nu' <$ or $> k_1/\epsilon\,\alpha_1$. The magnetic field has stabilizing effect and completely stabilizes certain wavenumber band $k > k^*$ (if $\nu' < k_1/\epsilon\,\alpha_1$), which was always unstable in the absence of magnetic field, for the potentially unstable arrangement. However, the system is always unstable for the wave number band $k < k^*$ or if $\nu' > k_1/\epsilon\,\alpha_1$

Keywords Instability of superposed fluids, viscoelasticity, porous medium, hydromagnetics

PACS Nos. 47.20.Ma, 47.50.+d, 47.55.Mh

1. Introduction

The instability of the plane interface separating two viscous, Newtonian fluids when one is superposed over the other, under varying assumptions of hydrodynamics and hydromagnetics, has been studied by several authors and a detailed account of these investigations has been given by Chandrasekhar [1]. The stability of two superposed Walters' B′ viscoelastic fluids has been studied by Sharma and Kumar [2]. The medium has been considered to be non-porous in the above studies.

The stability of superposed viscoelastic fluids in porous medium has importance in chemical technology and industry. When a fluid flows through a porous medium, the gross effect is represented by the usual Darcy's law.

The present paper considers the instability of the plane interface between two superposed viscous and viscoelastic (Walters' B′) fluids in porous medium. The instability of the plane interface between two superposed viscous and viscoelastic (Walters' B′) fluids in porous medium is also considered when the fluids are electrically conducting and a variable horizontal magnetic field pervades the system.

*Corresponding Author

2. Formulation of the problem and perturbation equations

Consider a static state in which an incompressible Walters' B′ viscoelastic fluid is arranged in horizontal strata in porous medium and the pressure p and the density ρ are functions of the vertical coordinate z only. The character of the equilibrium of this initial state is determined, as usual, by supposing that the system is slightly disturbed and then by following its further evolution.

Let ρ, p and $\boldsymbol{q}$ denote respectively the density, the pressure and the filter velocity of the fluid. Let $\in$, k_1, μ, μ' and g stand for medium porosity, medium permeability, viscosity of fluid, viscoelasticity of fluid and acceleration due to gravity respectively. Then the equations of motion and continuity for the Walters' B′ viscoelastic fluid through porous medium (Sharma and Kumar [2]) are

$$\frac{\rho}{\in}\left[\frac{\partial \boldsymbol{q}}{\partial t}+\frac{1}{\in}(\boldsymbol{q}\cdot\nabla)\boldsymbol{q}\right] = -\nabla p-\rho g\boldsymbol{\lambda}-\frac{1}{k_1}\left(\mu-\mu'\frac{\partial}{\partial t}\right)\boldsymbol{q}, \quad (2.1)$$

$$\nabla\cdot\boldsymbol{q}=0. \quad (2.2)$$

Since the density of a fluid particle moving with the fluid remains unchanged, we have

$$\in \frac{\partial \rho}{\partial t}+(\boldsymbol{q}\cdot\nabla)\rho=0. \tag{2.3}$$

Let $\delta\rho, \delta p, \boldsymbol{q}(u,v,w)$ denote respectively the perturbations in density ρ, pressure p and fluid velocity (0, 0, 0). Then the linearized perturbation equations of Walters' (model B$'$) viscoelastic fluid layer are

$$\frac{\rho}{\in}\frac{\partial \boldsymbol{q}}{\partial t}=-\nabla\delta p-g\delta\rho\boldsymbol{\lambda} \quad \frac{1}{k_1}\left(\mu-\mu'\frac{\partial}{\partial t}\right. \tag{2.4}$$

$$\nabla\cdot\boldsymbol{q}=0, \tag{2.5}$$

$$\in\frac{\partial}{\partial t}(\delta\rho)=w(D\rho), \tag{2.6}$$

where $\boldsymbol{g}(0,0,-g)$ is the acceleration due to gravity, $\boldsymbol{\lambda}=(0,0,1)$ and $D=\frac{d}{dz}$.

Analyzing the disturbances into normal modes, we seek solutions whose dependence on x, y and t is given by

$$\exp\,(ik_x x+ik_y y+nt), \tag{2.7}$$

where n is, in general, a complex constant; k_x k_y are wave numbers along x- and y- directions and $k^2=k_x^2+k_y^2$.

For perturbations of the form (2 7), eqs (2.4) - (2.6) yield

$$\frac{\rho}{\in}nu=-ik_x\delta p-\frac{1}{k_1}(\mu-\mu'n)u, \tag{2.8}$$

$$\frac{\rho}{\in}nv=-ik_y\delta p-\frac{1}{k_1}(\mu-\mu'n)v, \tag{2.9}$$

$$\frac{\rho}{\in}nw=-D\delta p-g\delta\rho-\frac{1}{k_1}(\mu-\mu'n)w, \tag{2.10}$$

$$ik_x u+ik_y v+Dw=0, \tag{2.11}$$

$$\in n\delta\rho=-w(D\rho). \tag{2.12}$$

Eliminating δp between eqs.(2.8) – (2.10) and using eqs.(2.11) and (2.12), we obtain

$$\frac{n}{\in}\left[D(\rho Dw)-k^2\rho w\right]+\frac{1}{k_1}\left[D\{(\mu-\mu'n)Dw\}\right.$$

$$\left.-k^2(\mu-\mu'n)w\right]=\frac{-gk^2}{\in n}(D\rho)w \tag{2.13}$$

3. Two superposed viscous-viscoelastic fluids separated by a horizontal boundary

Here, we consider a uniform viscous fluid of density ρ_2 and viscosity μ_2, superposed over a viscoelastic (Walters' B$'$) fluid of density ρ_1, viscosity μ_1 and viscoelasticity μ_1', separated by a horizontal boundary at $z=0$. Then in each of the two regions of constant ρ, constant μ and constant lower μ', eq. (2.13) reduces to

$$(D^2-k^2)w=0. \tag{3.1}$$

The general solution of eq.(3.1) is

$$w=A\,e^{+kz}+Be^{-kz}, \tag{3.2}$$

where A and B are arbitrary constants.

The boundary conditions to be satisfied in present problem are

(i) The velocity $w\to 0$ when $z\to+\infty$ (for the upper fluid) and $z\to-\infty$ (for the lower fluid).

(ii) $w(z)$ is continuous at $z=0$.

(iii) The jump condition at the interface $z=0$ between the fluids. This is obtained by integrating eq.(2.13) across the interface at $z=0$ and is

$$\frac{n}{\in}(\rho_2 Dw_2-\rho_1 Dw_1)_{z=0}+\frac{1}{k_1}\left[\mu_2 Dw_2-(\mu_1-\mu_1'n)Dw_1\right]_{z=0}$$

$$\frac{-gk^2}{\in n}(\rho_2-\rho_1)w_0, \tag{3.3}$$

remembering the configuration that upper fluid is viscous (Newtonian) and lower fluid is viscoelastic (Walters' B$'$). Here, w_0 is the common value of w at $z=0$.

Applying the boundary conditions (i) and (ii) to the general solution (3.2), we can write

$$w_1=Ae^{kz},\quad (z<0), \tag{3.4}$$

$$Ae^{-kz},\quad (z>0), \tag{3.5}$$

where the same constant A has been chosen to ensure the continuity of w at $z=0$.

Applying the boundary condition (3.3) to the solutions (3.4) and (3.5), we obtain

$$\left(1-\frac{\in\nu'\alpha_1}{k_1}\right)n^2+\frac{\in\nu}{k_1}n-gk(\alpha_2-\alpha_1)=0, \tag{3.6}$$

where $\alpha_{1,2}=\frac{\rho_{1,2}}{\rho_1+\rho_2}$, $\nu_{1,2}=\frac{\mu_{1,2}}{\rho_{1,2}}$, and $\nu_1'=\frac{\mu_1'}{\rho_1}$

In deriving Eq. (3.6), we have made the assumption that the kinematic viscosities of the two fluids are equal *i.e.* $\nu_1=\nu_2=\nu$ and have put kinematic viscoelasticity $\nu_1'=\nu'$. However, this simplifying assumption does not obscure any of the essential features of the problem.

(a) Stable case ($\rho_2 < \rho_1$) :

For the potentially stable arrangement ($\rho_2 < \rho_1$) and $\nu' < \frac{k_1}{\in \alpha_1}$, all the coefficients of eq.(3.6) are positive. Therefore, both the roots of eq.(3.6) are either real and negative or complex conjugates with negative real parts. The system is thus stable in each case. However for the potentially stable arrangement ($\rho_2 < \rho_1$) and

$\frac{\quad}{\in \alpha_1}$, the coefficient of n^2 in eq.(3.6) is negative. There is a change of sign in the coefficients of eq.(3.6) and hence eq.(3.6) allows a positive root. The system is therefore unstable.

We thus conclude that for superposed viscous-viscoelastic (Walters' B′) fluids in porous medium and for the potentially stable arrangement, the system is stable or unstable according as $\nu' <$ or $> \frac{k_1}{\in \alpha_1}$. This is in contrast to the stability of Newtonian superposed fluids acted on by suspended particles in porous medium, where the system is always stable for the stable configuration [3].

(b) Unstable case ($\rho_2 > \rho_1$) :

For the potentially unstable arrangement ($\rho_2 > \rho_1$), the constant term in eq (3.6) is negative. Eq. (3.6) has a change of sign and hence allows one positive root. The occurrence of one positive root implies instability of the system. Thus for the potentially unstable case, the system is unstable for two superposed viscous-viscoelastic (Walters' B′) fluids in porous medium.

4. Effect of a variable horizontal magnetic field

Here, we consider the static state in which an incompressible, infinitely conducting Walters' (model B′) viscoelastic fluid is arranged in horizontal strata in porous medium in presence of a variable horizontal magnetic field $\boldsymbol{H}\,[H(z),0,0]$. Let $\boldsymbol{h}\,(h_x, h_y, h_z)$ denotes the perturbation in magnetic field and μ_e stands for magnetic permeability, then the linearized hydromagnetic perturbation equations, relevant to the problem, are

$$\frac{\rho}{\in}\frac{\partial \boldsymbol{q}}{\partial t} = -\nabla \delta p - g\delta\rho\boldsymbol{\lambda} - \frac{1}{k_1}\left(\mu - \mu'\frac{\partial}{\partial t}\right)\boldsymbol{q}$$

$$+\frac{\mu_e}{4\pi}\left[(\nabla\times\boldsymbol{h})\times\boldsymbol{H} + (\nabla\times\boldsymbol{H})\times\boldsymbol{h}\right], \tag{4.1}$$

$$\nabla\cdot\boldsymbol{h} = 0, \tag{4.2}$$

$$\in\frac{\partial \boldsymbol{h}}{\partial t} = \nabla\times(\boldsymbol{q}\times\boldsymbol{H}), \tag{4.3}$$

together with eqs.(2.2) and (2.3). For perturbation of the form (2.7), writing component equations and eliminating u, v, h_x, h_y, h_z, $\delta\rho$ and δp, eqs. (4.1) -(4.3) together with eqs.(2.2) - (2.3) yield

$$\frac{n}{\in}\left[D(\rho Dw) - k^2\rho w\right] + \frac{1}{k_1}\left[D\{(\mu - \mu' n)Dw\} - k^2(\mu - \mu' n)w\right]$$

$$+\frac{\mu_e k_1}{4\pi n \in}\left[D(H^2 Dw) - H^2k^2 w\right] = -\frac{gk^2}{\in n}(D\rho)w. \tag{4.4}$$

5. Two superposed viscous-viscoelastic fluids separated by a horizontal boundary in hydromagnetics

Here, we consider the case of two uniform fluids of densities, viscosities, viscoelasticities, magnetic fields as $\rho_2, \mu_2, \mu'_2 (= 0)$, H_2 (upper, viscous Newtonian fluid) and $\rho_1, \mu_1, \mu'_1, H_1$ (lower, viscoelastic Walters' B′ fluid), separated by a horizontal boundary at $z = 0$ in porous medium and is depicted in Figure 1. Then in each region of constant ρ, constant μ, constant μ', constant H, eq. (4.4) reduces to

$$\left(D^2 - k^2\right)w = 0 \tag{5.1}$$

The boundary conditions to be satisfied in the present problem are (i) the vanishing of w as $z \to +\infty$ and $z \to -\infty$, (ii) continuity of w at $z = 0$ and (iii) the jump condition. The jump condition is obtained by integrating eq.(4.4) across the interface at $z = 0$ and is

$$\frac{n}{\in}\left(\rho_2 Dw_2 - \rho_1 Dw_1\right)_{z=0} + \frac{1}{k_1}\left[\mu_2 Dw_2 - (\mu_1 - \mu'_1 n)Dw_1\right]_{z=0}$$

$$\frac{\mu_e k_1^2}{4\pi n \in}\left(H_2^2 Dw_2 - H_1^2 Dw_1\right)_{z=0}$$

$$\cdot\frac{gk^2}{\in n}\left(\rho_2 - \rho_1\right)w_0 \,. \tag{5.2}$$

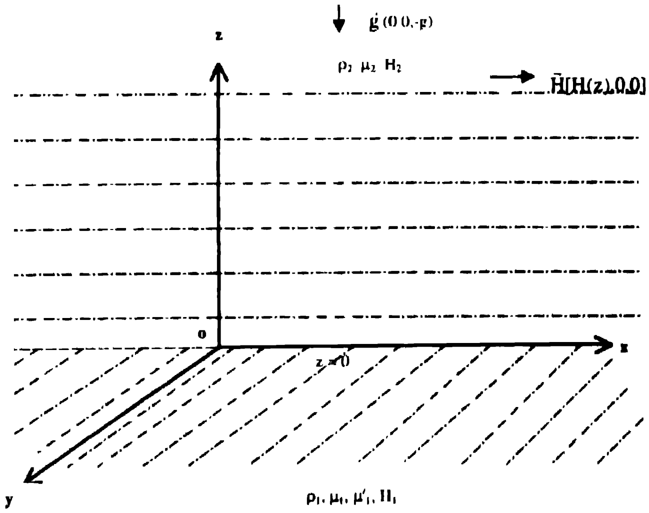

Figure 1. Two superposed viscous-viscoelastic fluids in hydromagnetics.

Applying the boundary conditions (i) and (ii), the solutions of eq.(5.1) are given by eqs. (3.4) and (3.5). Applying the condition (5.2) to the solutions (3.4) and (3.5), we obtain

$$\left(1-\frac{\in\nu'\alpha_1}{k}\right)n^2+\frac{\in\nu}{k}n+\left[2k_x^2V_A^2-gk(\alpha_2-\alpha_1)\right]=0, \quad (5.3)$$

where we have put

$$\alpha_{1,2}=\frac{\rho_{1,2}}{\rho_1+\rho_2},\ \nu_{1,2}=\frac{\mu_{1,2}}{\rho_{1,2}},\ \nu_1'=\frac{\mu_1'}{\rho_1},\ \nu_1=\nu_2=\nu$$

(kinematic viscosities of two fluids are assumed equal) and have put kinematic viscoelasticity $\nu_1'=\nu'$. The Alfvén velocities of the two fluids are assumed to be equal *i.e.*

$$V_A^2=\frac{\mu_e H_1^2}{4\pi(\rho_1+\rho_2)}=\frac{\mu_e H_2^2}{4\pi(\rho_1+\rho_2)}.$$

However, these simplifying assumptions do not obscure any of the essential features of the problem.

(a) Stable case ($\rho_2<\rho_1$) :

For the potentially stable arrangement ($\rho_2<\rho_1$) and $\nu'>\frac{}{\in\alpha_1}$ eq.(5.3) involves a change of sign and hence a positive root implying thereby that the system is unstable. However, for the potentially stable arrangement ($\rho_2<\rho_1$) and $\nu'<\frac{k_1}{\in\alpha_1}$, eq.(5.3) does not have a change of sign and so has no root with positive real part which means that the system is stable.

Thus, for superposed viscous-viscoelastic (Walters'B$'$) fluids in porous medium and for the potentially stable case, the system is stable or unstable according as $\nu'<\frac{k_1}{\in\alpha_1}$ or $\nu'>\frac{k_1}{\in\alpha_1}$ This is in contrast to the stability of Newtonian superposed fluids acted on by suspended particles in porous medium, where the system is always stable for the stable configuration [3].

(b) Unstable case ($\rho_2>\rho_1$) :

For the potentially unstable arrangement ($\rho_2>\rho_1$), and $\nu'<\frac{\cdot}{\in\alpha_1}$, the magnetic field has got a stabilizing effect and completely stabilizes the system for all wave numbers which satisfy the inequality

$$2k_x^2\,V_A^2>gk\,(\alpha_2-\alpha_1),$$

i.e. $\quad k>k^*$

$$\text{where } k^*=\frac{g(\alpha_2-\alpha_1)}{2V_A^2}\sec^2\theta, \quad (5.4)$$

and θ is the angle between $\boldsymbol{k}$ and magnetic field $\boldsymbol{H}$.

However, for the potentially unstable arrangement ($\rho_2>\rho_1$), the system is unstable for the wave number band $k<k^*$ or if $\nu'>\frac{}{\in\alpha_1}$

6. Conclusions

The stability of two superposed viscoelastic (Rivlin-Ericksen) fluids in porous medium, in hydrodynamics and hydromagnetics, has been studied by Sharma *et al* [4]. The system is found to be stable for potentially stable arrangement and unstable for the potentially unstable arrangement. The system is stable in hydromagnetics also for the potentially stable arrangement. However, for potentially unstable arrangement, the presence of magnetic field stabilizes the wave number band $k>k^*$ which was always unstable in the absence of magnetic field. The above findings are also true in two superposed Newtonian viscous fluids in porous medium. However, the results are different in Walters' B$'$ viscoelastic fluids.

This is in contrast to the stability of superposed viscous-viscoelastic (Walters' B$'$) fluids in porous medium in hydrodynamics and hydromagnetics. For the potentially unstable configuration and for two superposed viscous viscoelastic (Walters' B$'$) fluids in porous medium in hydrodynamics, the system is unstable for all wave numbers, whereas in hydromagnetics, the magnetic field has stabilizing effect if $\nu'<\frac{k_1}{\in\alpha_1}$ and completely stabilizes the wave number band $k>k^*$,

$$\text{where } k^*=\frac{g(\alpha_2-\alpha_1)}{2V_A^2}\sec^2\theta .$$

However, the system is always unstable if $\nu'>\frac{k_1}{\in\alpha_1}$. The system is also unstable for wave number band $k<k^*$.

For the potentially stable arrangement in hydrodynamics as well as in hydromagnetics, the system is stable or unstable according as ν' < or > $\frac{k_1}{\in\alpha_1}$ for two superposed viscous-viscoelastic (Walters'B$'$) fluids in porous medium.

References

[1] S Chandrasekhar *Hydrodynamic and Hydromagnetic Stability* (New York : Dover) (1981)

[2] R C Sharma and P Kumar *Czech J Phys* **47** 197 (1997)

[3] R C Sharma and P Kumar *J Math Phys. Sci* **29** 81 (1995)

[4] R C Sharma, P Kumar and Shivani Sharma *Indian J Phys* **75B** 337 (2001)